$H_2$ Robust Filtering with Optimality Gap Certification

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Outline

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   - Definition and example

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The notation is as follows:

- \( \mathbb{N} = \{1, \cdots, N\} \).
- \( \Lambda \) denotes the unitary simplex
  \[
  \Lambda = \left\{ \lambda \in \mathbb{R}^N : \sum_{i=1}^N \lambda_i = 1, \quad \lambda_i \geq 0 \right\}
  \]
- For matrices or transfer functions \( U_\lambda \) denotes the linear parameter dependence \( U_\lambda = \sum_{i=1}^N \lambda_i U_i \).
- Transfer functions are denoted as
  \[
  G(\zeta) = C(\zeta I - A)^{-1}B + D
  = \begin{bmatrix}
    A & B \\
    C & D
  \end{bmatrix}
  \]
  and \( G(\omega) \) represents either \( G(\zeta) \) calculated at \( \zeta = j\omega \) for continuous-time or \( G(\zeta) \) calculated at \( \zeta = e^{j\omega} \) for discrete-time systems.
The linear filtering design problem is as follows:

Filter transfer function \((\text{to be determined}) \rightarrow F(\omega)\).

System transfer function

\[
H(\omega) = \begin{bmatrix} T(\omega) \\ S(\omega) \end{bmatrix}
\]
Linear filtering

- The **classical** filter design problem is formally stated as

\[
\min_{F \in \mathcal{F}} J(F, H)
\]

where the transfer function \( H(\omega) \) is known.

- \( J(F, H) \) expresses the estimation error magnitude.
- \( \mathcal{F} \) imposes some desired characteristic to the optimal filter.

**a priori data and assumptions**

\[
\downarrow
\]

- Wiener filter (Kassam and Poor, 1985).
- Kalman filter (Anderson and Moore, 1979).
The robust filter design problem is formally stated as

$$\min_{F \in \mathcal{F}} \max_{H \in \mathcal{H}} J(F, H)$$

where $H(\omega)$ is not exactly known but the set $\mathcal{H}$ is given.

- Man-Nature game.
- The equilibrium solution $\implies$ how to determine?

**a priori data and assumptions**

- Robust Wiener filter (Poor, 1980).
- Robust Kalman filter (Jain, 1975).
Robust Wiener filter

- The **signal** is corrupted by an additive **noise** whose power spectral densities are $\Phi_S(\omega)$ and $\Phi_N(\omega)$, respectively. Collecting the densities in $\Phi(\omega)$, for the mean squared estimation error

$$J(F, \Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{|1 - F|^2 \Phi_S + |F|^2 \Phi_N\} \, d\omega$$

whenever $\mathcal{F}$ and $\mathcal{H}$ are convex sets, the equality holds

$$\max_{\Phi \in \mathcal{H}} \min_{F \in \mathcal{F}} J(F, \Phi) = \min_{F \in \mathcal{F}} \max_{\Phi \in \mathcal{H}} J(F, \Phi)$$

- $J(F, \Phi)$ admits a saddle point.
- The robust Wiener filter is a Wiener filter associated to the least favorable uncertainty.
Robust Kalman filter

- The $\mathcal{H}_2$ squared norm of the estimation error is

$$J(F, H) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr} \left( E(\omega)' E(\omega) \right) d\omega$$

where $E(\omega) = S(\omega) - F(\omega) T(\omega)$. For $\mathcal{H}$ composed by $H(\omega)$ with parameter uncertainty, the above functional does not present any relevant property. Hence

$$\max_{H \in \mathcal{H}} \min_{F \in \mathcal{F}} J(F, H) \leq \min_{F \in \mathcal{F}} \max_{H \in \mathcal{H}} J(F, H)$$

but, in general, the equality does not hold!

- The robust Kalman filter is not a Kalman filter associated to the least favorable uncertainty.
The parameter uncertainty to be considered is such that

$$\mathcal{H} = \begin{bmatrix} A_i & B_i \\ C_{yi} & D_{yi} \\ C_{zi} & D_{zi} \end{bmatrix}, \quad \forall i \in \mathbb{N}$$

Clearly, each $H \in \mathcal{H}$ has the state space representation

$$H(\lambda, \omega) = \begin{bmatrix} A_{\lambda} & B_{\lambda} \\ C_{y\lambda} & D_{y\lambda} \\ C_{z\lambda} & D_{z\lambda} \end{bmatrix}, \quad \lambda \in \Lambda$$

and it is important to keep in mind that the transfer function $H(\lambda, \omega)$ depends nonlinearly on $\lambda \in \Lambda$. 
The problem to be dealt with is stated as

\[ J^* = \min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} J(F, H(\lambda)) \]

where \( J(F, H(\lambda)) = \| S(\lambda, \omega) - F(\omega) T(\lambda, \omega) \|_2^2 \).

Assumptions:
- Each \( H \in \mathcal{H} \) is asymptotically stable.
- Each \( H \in \mathcal{H} \) is of invariant order \( n \).
In the general case, the $\mathcal{H}_2$ robust filter design problem is hard to solve. However, the following claims are important:

- The optimal filter $F^*$ is asymptotically stable.
- Is the optimal filter of full order? Unfortunately, the answer appears to be **NEGATIVE**. It is verified by means of some examples that, in general

\[ n^* > n \]

- If $(F^*, \lambda^*)$ is an equilibrium solution, then

\[ J(F^*, H(\lambda)) \leq J(F^*, H(\lambda^*)) = J^* \quad \forall \lambda \in \Lambda \]

defines the **minimum $\mathcal{H}_2$ guaranteed cost**.

\[ \downarrow \]

Paradigm for robust filtering
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Definition

The guaranteed cost strategy is based on the proposition of a convex functional $J_u(F)$ and a convex set $F_u$ such that

$$J(F, H(\lambda)) \leq J_u(F), \quad \forall \lambda \in \Lambda$$

$$F_u \subset F$$

yielding the $\mathcal{H}_2$ guaranteed cost design problem

$$\min_{F \in F_u} J_u(F)$$

presenting the following characteristics:

- Convex programming problem solvable with an LMI solver.
- Needs a posteriori performance qualification against the paradigm (unknown).
Since the inequality

\[ J(F^*, H(\lambda^*)) \leq \min_{F \in \mathcal{F}_u} J_u(F) = J_u(F^o) \]

holds, the main goal should be to reduce the gap between \( J_u(F^o) \) and \( J(F^*, H(\lambda^*)) \).

- At the present stage, to keep the problem convex, only full order filters \((n^* = n)\) are included in \( \mathcal{F}_u \) but we do not have any evidence that \( F^* \in \mathcal{F}_u \).

Whenever \( F^* \notin \mathcal{F}_u \), the gap may not be arbitrarily reduced.

- In general, the gap cannot be calculated since the paradigm \( J^* = J(F^*, H(\lambda^*)) \) is unknown. The quality of a particular guaranteed cost solution is tested against the ones available in the literature.
Example

- Consider a second-order continuous time system

\[
H(\omega) = \begin{bmatrix}
0 & -1 + 0.3\alpha & -2 \\
1 & -0.5 & 1 \\
-100 + 10\beta & 100 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

with two uncertain parameters satisfying

\[|\alpha| \leq 1, \ |\beta| \leq 1\]

The uncertain model is a polytope with \(N = 4\) vertices. Next, full order robust filters available in the literature are shown.
Example

\[ J_u \]

Shaked & Souza, 1995
Example

\[ J_u \]

Shaked & Souza, 1995

Geromel, 1999
Example

$J_u$

Shaked & Souza, 1995

Geromel, 1999

Souza & Trofino, 1999

$\text{time}$
Example

\[ J_u \]

- Shaked & Souza, 1995
- Geromel, 1999
- Souza & Trofino, 1999
- Tuan, Apkarian & Nguyen, 2001

\textbf{time}
Example

$J_u$

Shaked & Souza, 1995

Geromel, 1999

Souza & Trofino, 1999

Tuan, Apkarian & Nguyen, 2001

Geromel & Korogui, 2007
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Linear case

Consider the linear case characterized by

\[ H(\lambda, \omega) = H_\lambda(\omega) \]

or, in other words, \( H(\lambda, \omega) \) is linear in \( \lambda \in \Lambda \). Hence

\[
\mathcal{H} = \text{co} \left\{ \begin{bmatrix} A_i & B_i \\ C_{yi} & D_{yi} \\ C_{zi} & D_{zi} \end{bmatrix} H_i(\omega) \right\}, \quad \forall i \in \mathbb{N}
\]

implying that

\[
H_\lambda(\omega) = \begin{bmatrix} A & B(\lambda) \\ C_y & D_y(\lambda) \\ C_z & D_z(\lambda) \end{bmatrix}, \quad \lambda \in \Lambda
\]
Linear case

- The state space matrices are given by

\[
A = \text{diag}[A_1, \cdots, A_N], \quad B(\lambda)' = [\lambda_1 B'_1, \cdots, \lambda_N B'_N] \\
C_y = [C_{y1}, \cdots, C_{yN}], \quad D_y(\lambda) = \sum_{i=1}^{N} \lambda_i D_{yi} \\
C_z = [C_{z1}, \cdots, C_{zN}], \quad D_z(\lambda) = \sum_{i=1}^{N} \lambda_i D_{zi}
\]

where \(A, C_y\) and \(C_z\) are constant matrices.

**Fact (Gramian invariance)**

*In this particular (linear) case, the observability Gramian is constant over \(H\) \(\implies\) simple \(H_2\) norm calculation.*
Linear case

- The main consequence is that, in this particular case, the equilibrium solution of the robust filter paradigm can be exactly calculated.

**Theorem (Linear parameter dependent)**

The equilibrium solution of

$$\min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} \| S_\lambda(\omega) - F(\omega) T_\lambda(\omega) \|_2^2$$

**can be alternatively determined from**

$$\inf_{\sigma, \Upsilon} \{ \sigma : (\sigma, \Upsilon) \in \Psi \}$$

where the constraint $(\sigma, \Upsilon) \in \Psi$ is expressed by $N + 1$ LMIs. The optimal filter, of order $n \times N$, follows from the optimal value of $\Upsilon$. 

Linear case

- Putting aside poles and zeros cancellations, each $H_\lambda(\omega)$ is of order $n \times N$. Hence, the order of the optimal robust filter equals the maximum order of $H_\lambda(\omega)$ for all $\lambda \in \Lambda$.

- Many examples solved enable the conclusion that the order of the optimal robust filter $n^*$ satisfies

$$n \leq n^* \ll nN$$

- The optimal robust filter can be determined by any LMI solver.

- Linearity with respect to $\lambda \in \Lambda$ is essential to get the result.
**Example**

- Estimation of a signal corrupted by transmission noise.
  Convex combination of $N = 2$ **fourth order** transfer functions.
  Optimal **fourth order** filter associated to vertex “1”.
Optimal fourth order filter associated to vertex “2”.
Example

- Optimal robust **sixth order** filter, obtained after cancellation of two poles and zeros.
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Certification

- Recognizing that, in the general case of polytopic systems, the design problem

\[ J^* = \min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} J(F, H(\lambda)) \]

is very hard to solve, the strategy is to provide \( J_L \), \( J_H \) and a filter \( F_H(\omega) \) satisfying

\[ J_L \leq \min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} J(F, H(\lambda)) \leq \max_{\lambda \in \Lambda} J(F_H, H(\lambda)) \leq J_H \]

which certifies that:

- In terms of performance, the distance from \( F_H(\omega) \) to the paradigm \( F^*(\omega) \) is less than \( J_H - J_L \).
- The robust filter \( F_H(\omega) \) has a guaranteed cost less than \( J_H \).
The following inequality is the key issue to determine the lower bound $J_L$:

\[
J^* \geq \min_{F \in \mathcal{F}} \max_{i \in \mathbb{N}} J(F, H_i) \\
\geq \min_{F \in \mathcal{F}} \max_{i \in \mathbb{N}} \| S_i(\omega) - F(\omega) T_i(\omega) \|_2^2 \\
\geq \min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} \left\| \sum_{i=1}^{N} \lambda_i (S_i(\omega) - F(\omega) T_i(\omega)) \right\|_2^2 \\
\geq \min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} \left\| S_\lambda(\omega) - F(\omega) T_\lambda(\omega) \right\|_2^2
\]

A problem that we already know how to solve!
Using the previous theorem we obtain $J_L$ satisfying

$$J^* \geq J_L$$

and the associated $nN$-th order filter

$$F_L(\omega) = \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}$$

**Fact**

_The quantity $J_L$ is a lower bound to the equilibrium cost $J^*$. Consequently, any robust filter of any order has a performance limited below by $J_L$. Moreover, if the optimal $\lambda^* \in \Lambda$ is one vertex of $\Lambda$ then $J^* = J_L$. _
The simplest way to generate an upper bound is to define a subset $\mathcal{F}_H \subset \mathcal{F}$ yielding

$$\min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} J(F, H(\lambda)) \leq \min_{F \in \mathcal{F}_H} \max_{\lambda \in \Lambda} J(F, H(\lambda))$$

Clearly the choice of $\mathcal{F}_H$ is crucial. Hence, we consider $\mathcal{F}_H$ the set of all filters of the form

$$F_H(\omega) = \begin{bmatrix} A_L & B_L \\ C_H & D_H \end{bmatrix}$$

where $A_L$ and $B_L$ are the matrices of the optimistic filter $F_L(\omega)$. The matrices $C_H$ and $D_H$ are variables to be determined in order to minimize the upper bound $J_H$. 
Robust performance

- For this particular class of filters $\mathcal{F}_H$, the robust filter $F_H(\omega)$ and the associated upper bound ($\mathcal{H}_2$ guaranteed cost) $J_H$ can be calculated by any LMI solver.

Theorem (Robust performance)

A robust filter $F_H(\omega)$ and the associated upper bound $J_H$ satisfying

$$\min_{F \in \mathcal{F}_H} \max_{\lambda \in \Lambda} \| S(\lambda, \omega) - F(\omega) T(\lambda, \omega) \|_2^2 \leq J_H$$

can be alternatively determined from

$$\inf_{\sigma, \mathcal{X}} \{ \sigma : (\sigma, \mathcal{X}) \in \Omega \}$$

where the constraint $(\sigma, \mathcal{X}) \in \Omega$ is expressed by $2N$ LMIs. The optimal filter, of order $n \times N$, follows from the optimal value of $\mathcal{X}$. 
The design procedure can be stated as follows:

- **Lower bound determination**: Determine from a convex programming problem the lower bound and the associated filter

  \[ J_L, \quad F_L(\omega) \in \mathcal{F} \]

- **Upper bound determination**: Determine from a convex programming problem the upper bound and the associated robust filter

  \[ J_H, \quad F_H(\omega) \in \mathcal{F}_H \]

The quantity \( J_H - J_L \) is the optimality gap certification. It measures the distance from the robust filter \( F_H(\omega) \) to the paradigm \( F^*(\omega) \).
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Continuous and discrete-time

Continuous-time

- Second order systems with two uncertain parameters.

<table>
<thead>
<tr>
<th>$J_u$</th>
<th>$J_H$</th>
<th>$J_L$</th>
<th>Robust filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.382</td>
<td>2.1147</td>
<td>2.1144</td>
<td>$\frac{-1.1651(\zeta+1.55)}{(\zeta+280.1)(\zeta+0.1679)}$</td>
</tr>
<tr>
<td>2.382</td>
<td>2.1144</td>
<td>2.1144</td>
<td>$\frac{-1.163(\zeta+1.554)}{(\zeta+280)(\zeta+0.1679)}$</td>
</tr>
<tr>
<td>6.2846</td>
<td>6.2660</td>
<td>6.2451</td>
<td>$\frac{-0.053485(\zeta+15.88)}{(\zeta+212.2)(\zeta+0.3385)}$</td>
</tr>
<tr>
<td>93.365</td>
<td>15.8921</td>
<td>15.5091</td>
<td>$\frac{0.38817(\zeta-0.02347)}{(\zeta+0.2469)(\zeta+317.9)}$</td>
</tr>
<tr>
<td>100.963</td>
<td>13.7425</td>
<td>12.7187</td>
<td>$\frac{0.96248(\zeta+0.3906)(\zeta^2-0.02628\zeta+1.554)}{(\zeta+258.7)(\zeta+0.5124)(\zeta^2+0.8017\zeta+1.643)}$</td>
</tr>
</tbody>
</table>
Discrete-time

- Second order systems with one uncertain parameter.

<table>
<thead>
<tr>
<th></th>
<th>$J_u$</th>
<th>$J_H$</th>
<th>$J_L$</th>
<th>Robust filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>51.43</td>
<td>51.3809</td>
<td>51.3809</td>
<td></td>
<td>$-0.004905(\zeta-0.9193) \over (\zeta-1.246e-006)(\zeta-0.7722)$</td>
</tr>
<tr>
<td>56.60</td>
<td>56.0013</td>
<td>54.2035</td>
<td></td>
<td>$0.0018619(\zeta+4.735) \over (\zeta-2.213e-005)(\zeta+0.8507)$</td>
</tr>
<tr>
<td>1.887</td>
<td>1.8412</td>
<td>1.8412</td>
<td></td>
<td>$0.63939(\zeta-0.9224) \over (\zeta-0.3528)(\zeta-0.9079)$</td>
</tr>
<tr>
<td>0.6194</td>
<td>0.6153</td>
<td>0.6153</td>
<td></td>
<td>$-0.0090191(\zeta-0.7561) \over (\zeta-0.7722)$</td>
</tr>
<tr>
<td>0.3891</td>
<td>0.2533</td>
<td>0.2255</td>
<td></td>
<td>$-0.010193(\zeta+0.932) \over (\zeta+0.8506)$</td>
</tr>
<tr>
<td>—</td>
<td>0.6480</td>
<td>0.6480</td>
<td></td>
<td>$0.64807(\zeta-8.107e-006)(\zeta-0.9134) \over (\zeta-0.3528)(\zeta-0.9079)$</td>
</tr>
</tbody>
</table>
Practical application

The model for the displacement of a tapered bar with $\nu$ vibration modes and $m$ pairs of actuators / sensors co-located at positions $p_1, \cdots, p_m$ is

$$\ddot{x}_i(t) + \omega_i^2 x_i(t) = \sum_{j=1}^{m} \phi_i(p_j) u_j(t)$$

$$y_j(t) = \sum_{i=1}^{\nu} \phi_i(p_j)x_i(t)$$

where the natural frequencies $\omega_i$ and the eigenfunctions $\phi_i(\cdot)$ are known. The variable $u_j(t)$ is the intensity of the force produced by each actuator.
It is assumed that:

- The velocity and displacement measurements produced by each sensor are corrupted by orthogonal noises $w_{1j}(t)$ and $w_{2j}(t)$ with unitary intensities.
- The force intensity is $u_j(t) = -\sum_{k=1}^{m} \delta R_{jk}(\dot{y}_k(t) + w_{1k}(t))$, where $R \in \mathbb{R}^{m \times m}$ is a symmetric and positive definite matrix chosen in order to impose to the closed loop system a pre-specified nominal performance. The scalar $\delta \in \mathbb{R}$ represents the uncertainty on the gains of the force actuators.

The goal: Design a robust filter for the estimation of the displacement of the bar at positions $l_1, \ldots, l_r$ taking into account that the uncertain parameter $\delta \in \mathbb{R}$ belongs to the interval $|\delta - 1| \leq 0.5$. This corresponds to 50% parameter deviation around the nominal value.
Practical application

- The final model is of the form

\[
H(\omega) = \begin{bmatrix}
0 & I & 0 & 0 \\
-\Pi & -\delta BRB' & -\delta BR & 0 \\
C & 0 & 0 & I \\
M & 0 & 0 & 0
\end{bmatrix}
\]

where \( \Pi = \text{diag}(\omega_1^2, \ldots, \omega_{2\nu}^2) \), \( B_{ij} = \phi_i(p_j) \), \( C_{ji} = \phi_i(p_j) \) and \( M_{ji} = \phi_i(l_j) \). This transfer function represents a linear time invariant system of order \( n = 2\nu \) with \( 2m \) inputs and \( m + r \) outputs.

- For simulation purpose we have considered \( m = 2, r = 1 \) and \( n \in \{2, 4, 6, 8\} \).
The table below gives the performance of each robust filter

<table>
<thead>
<tr>
<th>n</th>
<th>$J_H$</th>
<th>$\max_{\lambda \in \Lambda} J(F_H, H(\lambda))$</th>
<th>$J_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1565</td>
<td>0.1565</td>
<td>0.1565</td>
</tr>
<tr>
<td>4</td>
<td>0.4485</td>
<td>0.4459</td>
<td>0.4450</td>
</tr>
<tr>
<td>6</td>
<td>0.5320</td>
<td>0.5254</td>
<td>0.5232</td>
</tr>
<tr>
<td>8</td>
<td>0.5714</td>
<td>0.5616</td>
<td>0.5584</td>
</tr>
</tbody>
</table>

The following conclusions can be drawn:

- In all cases, the optimality gap $J_H - J_L$ is small (less than 3%).
- The third column provides the guaranteed cost, calculated by gridding, associated to the robust filter $F_H(\omega)$. Within the precision level (1%), we can say that in each case the $\mathcal{H}_2$ (almost) optimal robust filter has been determined.
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Conclusion

- An open problem: Generalization to cope with $\mathcal{H}_\infty$-norm.

\[ \min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} \| S_\lambda(\omega) - F(\omega)T_\lambda(\omega) \|_\infty^2 \]

by means of LMIs is not a simple task!
Conclusion

Thank you for your attention!